AP Calculus BC Module 4C
Linearization and L'Hospital's Rule

Unit 4: Contextual Applications of Differentiation, Sections 4.6–4.7
Topics: approximating a function value with its tangent line (local linearization); deciding whether that approximation over- or under-estimates the true value using concavity; and evaluating limits of the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$ with L'Hospital's Rule

Learning Goals

By the end of this module, you should be able to:

- write the **linearization** (the tangent-line, or local linear, approximation) $L(x) = f(a)+f'(a)(x-a)$ of a function at a point, and use it to approximate values of f near $x = a$;
- decide whether a tangent-line approximation is an **overestimate** or an **underestimate** of the true value by reasoning about the *concavity* of f (the sign of f'') near the point of tangency;
- recognize when a limit has **indeterminate form** $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and apply **L'Hospital's Rule** correctly, differentiating the numerator and denominator *separately*;
- read a linearization *backwards* from a table or a graph: build the tangent-line estimate from given values of $f(a)$ and $f'(a)$ (or from a slope read off a graph) and interpret it;
- BC recognize the linearization $L(x) = f(a) + f'(a)(x - a)$ as the **first-degree Taylor polynomial** of f at a: the same point-and-slope idea that Unit 10 extends to higher-degree Taylor polynomials and a sharp error bound;
- state every approximation and every limit *in context*, with correct units where applicable, and with a complete justification that names the derivative doing the work (the tangent slope $f'(a)$, the sign of f'', or the indeterminate form that licenses L'Hospital's Rule).

Note on scope and BC weight (Unit 4). Both ideas in this module also live in AB, and Unit 4 is a lighter slice of the BC exam (about 6–9% of the multiple-choice), so we keep the intuition and the core routines and move efficiently rather than re-padding. Two supporting ideas are developed elsewhere and appear here only as tools: *concavity* (the sign of f'' and what it says about the shape of a graph) is formalized in Module 5C; here we use just the picture that a graph curving upward lies above its tangent lines and a graph curving downward lies below them. *Point-slope form* and basic differentiation rules are assumed from earlier units. For L'Hospital's Rule, the AP exam (both AB and BC) assesses only the two ratio forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$; other indeterminate forms exist but are not tested, so we keep to those two. The Fall 2026 CED clarifications touch only Units 5 and 7 and leave this module unchanged. One forward link is worth flagging for BC: the linearization is exactly the degree-one Taylor polynomial, and the concavity over/under argument is the sign of the next Taylor term, both of which Unit 10 makes precise.

Condensed Lecture

This module collects two exam-favorite tools that both grow out of a single fact you already know: near a point, a smooth curve looks almost exactly like its tangent line. That observation lets you *approximate* hard-to-compute values (linearization), and it explains *why* comparing two derivatives can resolve a limit that looks like nonsense (L'Hospital's Rule). Here is the whole toolkit in brief; the Full Lecture explains the *why* behind each piece.

Linearization: the tangent line as a local approximation

Zoom in far enough on a differentiable curve and it becomes indistinguishable from its tangent line at that point. So the tangent line is a ready-made approximation of f for inputs near the point of tangency.

The **linearization** of f at $x = a$ is the tangent-line function

$$L(x) = f(a) + f'(a)(x - a).$$

For inputs x near a, $f(x) \approx L(x)$. To approximate f at a nearby input b, evaluate $L(b) = f(a) + f'(a)(b - a)$, choosing a to be a convenient point where $f(a)$ and $f'(a)$ are easy to compute exactly.

BC **Looking ahead.** This linearization is the *first-degree Taylor polynomial* of f at a: the polynomial that matches f's value and slope at a. In Unit 10 you will add more terms, $\frac{f''(a)}{2}(x-a)^2$ and beyond, to get sharper approximations, and the sign of that first added term is exactly the concavity argument used below for over/underestimates.

Overestimate or underestimate? Let concavity decide

The tangent line is a straight stand-in for a curved graph, so whether it sits above or below the true curve depends on which way the curve bends near $x = a$.

If f is **concave up** near a (so $f'' > 0$), the graph lies *above* its tangent line, and the approximation $L(b)$ is an **underestimate** of $f(b)$.
If f is **concave down** near a (so $f'' < 0$), the graph lies *below* its tangent line, and the approximation $L(b)$ is an **overestimate** of $f(b)$.

Indeterminate forms and L'Hospital's Rule

Some limits of a ratio give the meaningless-looking shapes $\frac{0}{0}$ or $\frac{\infty}{\infty}$. These are called *indeterminate* because the answer is not decided by the form alone; it depends on *how fast* the top and bottom approach their limits, which is exactly what their derivatives measure.

L'Hospital's Rule. Suppose f and g are differentiable near a with $g'(x) \neq 0$ near a. If

$$\lim_{x \to a} \frac{f(x)}{g(x)} \text{ has indeterminate form } \frac{0}{0} \text{ or } \frac{\infty}{\infty},$$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided the limit on the right exists (or is $\pm\infty$). The rule also holds for one-sided limits and for $x \to \pm\infty$.

Justifying your answer

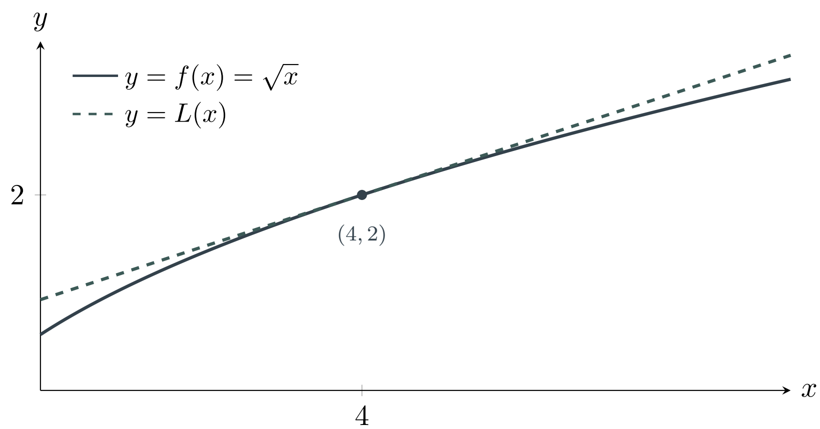
On the AP Exam, an approximation earns full credit only when the work names the tangent slope it used, and an over/under claim earns credit only when it points to the concavity (the sign of f''). A L'Hospital answer earns credit only when you first state that the limit has form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. The exact sentences readers reward appear in the Full Lecture.

Full Lecture

Why a tangent line approximates: local linearity

Take any differentiable function and zoom in toward one of its points. The more you magnify, the flatter and straighter the graph looks, until at high enough magnification it is visually a straight line, and that line is exactly the tangent. This is what "differentiable" buys you: *local linearity.* So if you want f at an input close to a point a where you already know $f(a)$ and the slope $f'(a)$, you can ride along the tangent line instead of the curve and barely miss.

Here is the picture for $f(x) = \sqrt{x}$ near $a = 4$. We know $f(4) = 2$ exactly, and the slope there is $f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$. The tangent line at $(4, 2)$ is $L(x) = 2 + \frac{1}{4}(x - 4)$. Near $x = 4$ the line and the curve are nearly on top of each other.



To approximate $\sqrt{4.1}$, feed $x = 4.1$ into the line, not the curve:

$$\sqrt{4.1} \approx L(4.1) = 2 + \tfrac{1}{4}(4.1 - 4) = 2 + \tfrac{1}{4}(0.1) = 2.025.$$

The true value is $\sqrt{4.1} = 2.024846\ldots$, so the tangent line missed by less than a thousandth. That is the entire idea; we now make it a formula.

The **linearization** of a differentiable function f at $x = a$ is

$$L(x) = f(a) + f'(a)(x - a),$$

the equation of the tangent line at $\big(a, f(a)\big)$ written in point-slope form. For x near a,

$$f(x) \approx L(x) = f(a) + f'(a)(x - a).$$

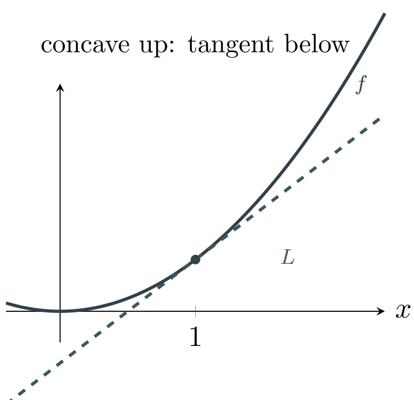
Choose a to be the nearest convenient input where $f(a)$ and $f'(a)$ are exact (a perfect square, a known angle, $x = 0$, and so on).

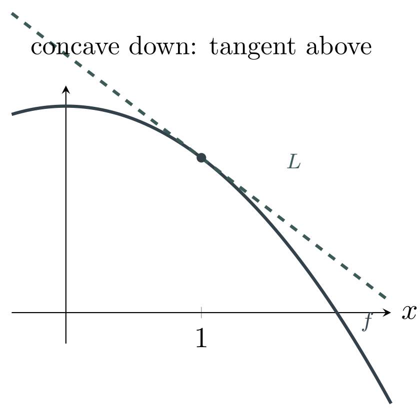
BC **The linearization is a Taylor polynomial.** The line $L(x) = f(a) + f'(a)(x - a)$ is built to match f in two ways at $x = a$: the constant term $f(a)$ makes the values agree, and the slope term $f'(a)$ makes the first derivatives agree. That is precisely the recipe for the *first-degree Taylor polynomial* $T_1(x)$ centered at a. For $f(x) = \sqrt{x}$ at $a = 4$, expanding gives $T_1(x) = 2 + \frac{1}{4}(x - 4)$, the same line you just drew. In Unit 10 you keep matching higher derivatives, adding $\frac{f''(a)}{2!}(x-a)^2$, then $\frac{f'''(a)}{3!}(x-a)^3$, and so on, and you bound the error with the Lagrange remainder. Everything in this module is the degree-one case of that bigger story.

Trap: the base point a is where you *know* the function, not the value you want. To estimate $\sqrt{4.1}$, the base point is $a = 4$ (a perfect square), and the target is $b = 4.1$. Students sometimes plug the target into f' or forget the $(x - a)$ factor and just write $f(a) + f'(a)$. Keep the structure intact: *value at the base, plus slope at the base, times the small step* $(b - a)$.

Overestimate or underestimate: read the bend

A tangent line is straight, but the graph it approximates is curved, so the line and the curve drift apart as you move away from the point of tangency. *Which way* they drift is set by the concavity. If the graph curves upward (concave up), it bows up and away from the straight tangent line, so the curve is *above* the line and the line *underestimates*. If the graph curves downward (concave down), it bows below the tangent, so the line *overestimates*. The two pictures below show both cases.





Suppose L is the linearization of f at $x = a$, used to approximate $f(b)$ for b near a.

- If f is **concave up** near a (that is, $f'' > 0$ there), the graph lies above the tangent line, so $L(b) \leq f(b)$: the approximation is an **underestimate**.
- If f is **concave down** near a (that is, $f'' < 0$ there), the graph lies below the tangent line, so $L(b) \geq f(b)$: the approximation is an **overestimate**.

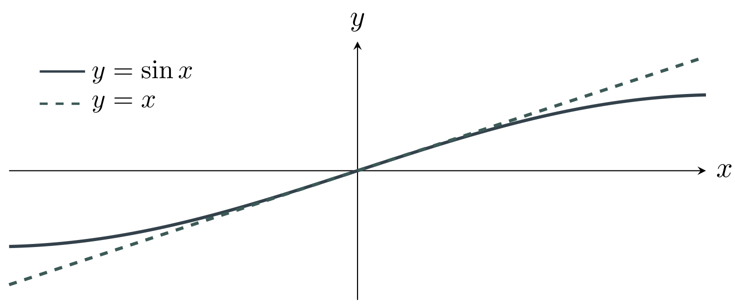
In the $\sqrt{x}$ example above, $f'(x) = \frac{1}{2}x^{-1/2}$ gives $f''(x) = -\frac{1}{4}x^{-3/2} < 0$ for $x > 0$, so $\sqrt{x}$ is concave down and the estimate 2.025 is an overestimate of $\sqrt{4.1} = 2.0248\ldots$, exactly as we found.

Trap: over/under comes from concavity, not from whether the function is increasing. A function can be increasing and concave down (like $\sqrt{x}$) or increasing and concave up. What decides the tangent line's position is the *sign of* f'', not the sign of f'. Always justify an over/under claim by naming the concavity, never by saying "the function is going up."

Indeterminate forms: why $\frac{0}{0}$ is genuinely undecided

Suppose $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$. What is $\lim_{x\to a} \dfrac{f(x)}{g(x)}$? You cannot tell from the zeros alone. If the numerator rushes to zero much faster than the denominator, the ratio collapses to 0; if the denominator wins the race, the ratio blows up; if they go at comparable speeds, the ratio settles on a finite nonzero number. The form $\frac{0}{0}$ does not pick a winner, which is why it is called *indeterminate*. "Speed toward zero" is precisely a derivative, so the resolution should involve f' and g', and it does.

A clean way to see this: near $x = 0$, $\sin x$ and x both head to 0, and the graphs hug each other so closely that $\dfrac{\sin x}{x} \to 1$. Their slopes at 0 are $\cos 0 = 1$ and 1, and the ratio of slopes is $\frac{1}{1} = 1$, matching the limit.



The same logic works for $\frac{\infty}{\infty}$: if both top and bottom run off to infinity, the limit of the ratio is decided by how fast each grows, again a derivative comparison. L'Hospital's Rule turns this intuition into a tool.

L'Hospital's Rule. Let f and g be differentiable on an open interval containing a (except possibly at a), with $g'(x) \neq 0$ near a. If

$$\lim_{x\to a} \frac{f(x)}{g(x)} \quad \text{has the indeterminate form} \quad \frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty},$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

as long as the limit on the right exists or is $\pm\infty$. The statement also holds for one-sided limits and for $x \to +\infty$ or $x \to -\infty$.

Trap: L'Hospital differentiates top and bottom *separately*; it is *not* the quotient rule. You replace $\frac{f}{g}$ with $\frac{f'}{g'}$, differentiating the numerator and the denominator on their own. Do not apply the quotient rule to $\frac{f}{g}$. And you must *first* confirm the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$: applying L'Hospital to a limit that is not indeterminate (say $\frac{2}{5}$) gives a wrong answer.

Trap: re-check the form before each new application. One pass of L'Hospital can leave you with another $\frac{0}{0}$ or $\frac{\infty}{\infty}$. When it does, you may apply the rule again, but only after confirming the new ratio is still indeterminate. Once the ratio is no longer indeterminate, stop and evaluate by substitution.

Writing the answer (AP language)

On the AP Exam, an approximation or a limit earns full credit only when it is justified by naming the right derivative. The templates below are the exact sentences readers reward.

Linearization: "The linearization of f at $x = a$ is $L(x) = f(a)+f'(a)(x-a)$, so $f(b) \approx L(b) =$ *(value)*, using the tangent slope $f'(a) =$ *(value)*."

Over/underestimate: "The approximation is an *(overestimate/underestimate)* because f is *(concave down/concave up)* near $x = a$ (that is, f'' is *(negative/positive)* there), so the tangent line lies *(above/below)* the graph."

L'Hospital's Rule: "As $x \to a$ the limit has indeterminate form *($\frac{0}{0}$ / $\frac{\infty}{\infty}$)*, so by L'Hospital's Rule $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} =$ *(value)*."

Using the calculator (when the exam allows it)

On calculator-active questions the AP exam expects only four built-in operations: (1) graph a function, (2) find a zero or intersection, (3) compute a numerical derivative at a point, and (4) compute a definite integral. For this module the useful ones are:

- **Getting the tangent slope.** When $f'(a)$ is awkward to differentiate by hand, use the numerical-derivative feature to evaluate $f'(a)$ at the base point, then build $L(x) = f(a) + f'(a)(x - a)$ and report $L(b)$. Write the slope down: a bare estimate earns little.
- **Seeing the approximation.** Graphing $y = f(x)$ together with $y = L(x)$ shows the line hugging the curve near a and confirms visually whether the line sits above (overestimate) or below (underestimate) the graph.

Calculator note for L'Hospital. L'Hospital problems are almost always on the *no-calculator* section: you are expected to show the differentiation of the numerator and denominator and the re-checking of the form by hand. A numerical answer with no work earns little here.

Model Problem 1: Linearization and the Over/Under Question

Let $f(x) = \sqrt{x}$.

(a) Find the linearization $L(x)$ of f at $a = 25$.

(b) Use L to approximate $\sqrt{25.3}$.

(c) Is the approximation an overestimate or an underestimate of $\sqrt{25.3}$? Justify your answer.

Solution

(a) The linearization at $a = 25$.

> The base point $a = 25$ is a perfect square, so $f(25) = \sqrt{25} = 5$ is exact. For the slope, differentiate $f(x) = \sqrt{x} = x^{1/2}$:
>
> $$f'(x) = \tfrac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}},$$
>
> so $f'(25) = \dfrac{1}{2\sqrt{25}} = \dfrac{1}{10}$. The linearization is the tangent line at $(25, 5)$:
>
> $$L(x) = f(25) + f'(25)(x - 25) = 5 + \tfrac{1}{10}(x - 25).$$

(b) Approximating $\sqrt{25.3}$.

> The target input is $b = 25.3$, a small step of 0.3 past the base point. Substituting into L from part (a),
>
> $$\sqrt{25.3} \approx L(25.3) = 5 + \tfrac{1}{10}(25.3 - 25) = 5 + \tfrac{1}{10}(0.3) = 5.03.$$

(c) Overestimate or underestimate.

> The sign of the second derivative settles this. Differentiate $f'(x) = \frac{1}{2}x^{-1/2}$:
>
> $$f''(x) = -\tfrac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt{x^3}}.$$
>
> For $x > 0$ this is negative, so f is concave down near $a = 25$. A concave-down graph lies below its tangent line, so the tangent-line value 5.03 is an **overestimate** of $\sqrt{25.3}$. (Indeed $\sqrt{25.3} = 5.029910\ldots$, just below 5.03.)

Model Problem 2: L'Hospital's Rule on Both Indeterminate Forms

Evaluate each limit, showing that L'Hospital's Rule applies at each step.

(a) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$

(b) $\lim_{x \to \infty} \frac{3x^2 - 5x}{2x^2 + 7}$

Solution

(a) A $\frac{0}{0}$ form that needs two passes.

As $x \to 0$, the numerator $1 - \cos x \to 1 - 1 = 0$ and the denominator $x^2 \to 0$, so the limit has indeterminate form $\frac{0}{0}$ and L'Hospital's Rule applies. Differentiate the top and bottom *separately*:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x}.$$

Check the new ratio: as $x \to 0$, $\sin x \to 0$ and $2x \to 0$, so it is again $\frac{0}{0}$ and we apply the rule a second time:

$$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{\cos 0}{2} = \frac{1}{2}.$$

The ratio $\frac{\cos x}{2}$ is no longer indeterminate, so we stop and substitute. The limit is $\frac{1}{2}$.

(b) An $\frac{\infty}{\infty}$ form.

As $x \to \infty$, the numerator $3x^2 - 5x \to \infty$ and the denominator $2x^2 + 7 \to \infty$, so the limit has indeterminate form $\frac{\infty}{\infty}$ and L'Hospital's Rule applies. Differentiating top and bottom separately,

$$\lim_{x \to \infty} \frac{3x^2 - 5x}{2x^2 + 7} = \lim_{x \to \infty} \frac{6x - 5}{4x}.$$

This is still $\frac{\infty}{\infty}$, so apply the rule once more:

$$\lim_{x \to \infty} \frac{6x - 5}{4x} = \lim_{x \to \infty} \frac{6}{4} = \frac{3}{2}.$$

The limit is $\frac{3}{2}$. (This matches the shortcut for a ratio of polynomials of equal degree: the limit is the ratio of leading coefficients, $\frac{3}{2}$.)

Model Problem 3: A Linearization in Context

Water is being pumped into a tank. The volume $V(t)$ of water in the tank, in liters, is a differentiable function of time t in minutes. At $t = 10$ minutes the tank holds $V(10) = 50$ liters and the water is flowing in at a rate of $V'(10) = 4$ liters per minute. For times t near 10, the graph of V is concave down.

(a) Use a tangent-line approximation to estimate the volume of water in the tank at $t = 10.5$ minutes.

(b) Is your estimate an overestimate or an underestimate of the true volume $V(10.5)$? Justify your answer.

Solution

(a) Estimating $V(10.5)$.

> The base point is $a = 10$, where both the value and the rate are given, so no differentiation is needed: $V(10) = 50$ and $V'(10) = 4$. The linearization is
>
> $$L(t) = V(10) + V'(10)\,(t - 10) = 50 + 4\,(t - 10).$$
>
> At $t = 10.5$, a step of 0.5 minute past the base point,
>
> $$V(10.5) \approx L(10.5) = 50 + 4(10.5 - 10) = 50 + 4(0.5) = 52 \text{ liters.}$$
>
> So about 52 liters of water are in the tank at $t = 10.5$ minutes.

(b) Overestimate or underestimate.

> The problem states that V is concave down near $t = 10$, so $V'' < 0$ there and the graph lies below its tangent line. The tangent-line estimate therefore sits above the true curve, making 52 liters an **overestimate** of the true volume $V(10.5)$. In context: because the inflow is slowing (concave down), the constant-rate tangent line predicts a bit more water than the tank actually gains.

Guided Problem Solving

Guided 1: Building a linearization from a known point

Let $f(x) = \ln x$. Using the linearization of f at $a = 1$, the best approximation of $\ln(1.1)$ is

(A) 0

(B) 0.1

(C) 1.1

(D) 0.0953

(E) $\ln(1.1)$ cannot be approximated this way.

Solution. At the base point $a = 1$, $f(1) = \ln 1 = 0$. Differentiate $f(x) = \ln x$:

$$f'(x) = \frac{1}{x}, \qquad f'(1) = 1.$$

The linearization is $L(x) = 0 + 1\,(x - 1) = x - 1$, so $\ln(1.1) \approx L(1.1) = 1.1 - 1 = 0.1$. (The true value is $0.0953\ldots$; choice (D) is the exact value, not the tangent-line estimate the question asks for.) The answer is **(B)**.

Guided 2 (FRQ): Full linearization with a decreasing function

Let $f(x) = \dfrac{1}{x}$.

(a) Find the linearization $L(x)$ of f at $a = 2$.

(b) Use L to approximate $\dfrac{1}{2.1}$.

(c) Determine whether the approximation is an overestimate or an underestimate, with justification.

Solution.

(a) The linearization at $a = 2$.

> At the base point, $f(2) = \frac{1}{2}$. Differentiate $f(x) = x^{-1}$:
>
> $$f'(x) = -x^{-2} = -\frac{1}{x^2}, \qquad f'(2) = -\frac{1}{4}.$$
>
> The linearization is
>
> $$L(x) = f(2) + f'(2)(x - 2) = \tfrac{1}{2} - \tfrac{1}{4}(x - 2).$$

(b) Approximating $\frac{1}{2.1}$.

With target $b = 2.1$, a step of 0.1,

$$\frac{1}{2.1} \approx L(2.1) = \tfrac{1}{2} - \tfrac{1}{4}(2.1 - 2) = 0.5 - 0.25(0.1) = 0.475.$$

(c) Overestimate or underestimate.

Differentiate $f'(x) = -x^{-2}$ once more:

$$f''(x) = 2x^{-3} = \frac{2}{x^3}.$$

For $x > 0$ this is positive, so f is concave up near $a = 2$ and the graph lies above its tangent line. The approximation 0.475 is therefore an **underestimate**. (The true value is $\frac{1}{2.1} = 0.47619\ldots$, just above 0.475.)

Guided 3: A single application of L'Hospital's Rule

$\displaystyle\lim_{x\to 0} \frac{e^{2x} - 1}{x} =$

(A) 0

(B) 1

(C) 2

(D) $\frac{1}{2}$

(E) the limit does not exist

Solution. As $x \to 0$, the numerator $e^{2x} - 1 \to e^0 - 1 = 0$ and the denominator $x \to 0$, so the form is $\frac{0}{0}$ and L'Hospital's Rule applies. Differentiating top and bottom separately (chain rule on e^{2x}),

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x} = \lim_{x\to 0} \frac{2e^{2x}}{1} = 2e^0 = 2.$$

The answer is **(C)**.

Guided 4: Checking the form before applying the rule

For which of the following limits is L'Hospital's Rule *directly* applicable?

(A) $\displaystyle\lim_{x\to 0} \frac{x+2}{x+1}$

(B) $\displaystyle\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$

(C) $\lim_{x \to 0} \frac{\cos x}{x}$

(D) $\lim_{x \to 3} \frac{x-3}{x+3}$

(E) $\lim_{x \to 0} \frac{x}{x+4}$

Solution. L'Hospital's Rule applies only to the indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, so test each by substitution. (A) gives $\frac{2}{1}$, (C) gives $\frac{1}{0}$ (infinite, not indeterminate), (D) gives $\frac{0}{6} = 0$, and (E) gives $\frac{0}{4} = 0$; none of these is indeterminate. Only (B) gives $\frac{1-1}{1-1} = \frac{0}{0}$, so the rule applies directly there (it yields $\lim_{x \to 1} \frac{2x}{1} = 2$). The answer is **(B)**.

Guided 5 (FRQ): Two limits, two forms

Evaluate each limit, citing the indeterminate form at each step.

(a) $\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}}$

(b) $\lim_{x \to 0} \frac{1 - \cos(2x)}{x^2}$

Solution.

(a) An $\frac{\infty}{\infty}$ form.

As $x \to \infty$, $\ln x \to \infty$ and $\sqrt{x} \to \infty$, so the form is $\frac{\infty}{\infty}$ and L'Hospital's Rule applies. Differentiate top and bottom ($\sqrt{x} = x^{1/2}$):

$$\lim_{x \to \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{\frac{1}{2}x^{-1/2}}.$$

Simplify the compound fraction by multiplying through:

$$\frac{1/x}{\frac{1}{2}x^{-1/2}} = \frac{1}{x} \cdot \frac{2}{x^{-1/2}} = \frac{2x^{1/2}}{x} = \frac{2}{\sqrt{x}}.$$

As $x \to \infty$, $\frac{2}{\sqrt{x}} \to 0$. The limit is 0. (The denominator grows faster, so the ratio vanishes.)

(b) A $\frac{0}{0}$ form needing two passes.

As $x \to 0$, $1 - \cos(2x) \to 1 - 1 = 0$ and $x^2 \to 0$, so the form is $\frac{0}{0}$. Applying the rule (chain rule on $\cos(2x)$):

$$\lim_{x \to 0} \frac{1 - \cos(2x)}{x^2} = \lim_{x \to 0} \frac{2\sin(2x)}{2x} = \lim_{x \to 0} \frac{\sin(2x)}{x}.$$

This is again $\frac{0}{0}$, so apply the rule once more:

$$\lim_{x \to 0} \frac{\sin(2x)}{x} = \lim_{x \to 0} \frac{2\cos(2x)}{1} = 2\cos 0 = 2.$$

The limit is 2.

Guided 6 (FRQ): A contextual estimate read from given data

A cooling cup of coffee has temperature $C(t)$, in degrees Celsius, that is a differentiable function of time t in minutes. At $t = 4$ minutes, $C(4) = 70$ and $C'(4) = -3$. For times near $t = 4$, the graph of C is concave up.

(a) Write the tangent-line approximation to C at $t = 4$ and use it to estimate the temperature at $t = 4.5$ minutes.

(b) Interpret $C'(4) = -3$ in the context of this problem, with units.

(c) Is your estimate from part (a) an overestimate or an underestimate of $C(4.5)$? Justify your answer.

Solution.

(a) Estimating $C(4.5)$.

> The base point is $t = 4$, where the value and rate are given. The linearization is
>
> $$L(t) = C(4) + C'(4)(t - 4) = 70 - 3\,(t - 4).$$
>
> At $t = 4.5$, a step of 0.5 minute,
>
> $$C(4.5) \approx L(4.5) = 70 - 3(4.5 - 4) = 70 - 3(0.5) = 68.5\ ^\circ\text{C}.$$

(b) Interpreting $C'(4)$.

> $C'(4) = -3$ is the instantaneous rate of change of temperature at $t = 4$ minutes: the coffee is cooling at 3 degrees Celsius per minute at that instant. (The negative sign means the temperature is falling.)

(c) Overestimate or underestimate.

> Near $t = 4$ the graph of C is concave up ($C'' > 0$), so it lies above its tangent line. The tangent-line estimate therefore sits below the true curve, making $68.5\,^\circ$C an **underestimate** of $C(4.5)$. In context: the cooling is slowing down, so the temperature does not drop quite as far as the constant-rate tangent line predicts.

Independent Problem Solving

Independent 1: A tangent-line estimate

Using the linearization of $f(x) = \sqrt{x}$ at $a = 9$, the approximation of $\sqrt{9.2}$ is

(A) 3

(B) 3.0333

(C) 3.2

(D) 3.0331

(E) 9.0333

Solution. At $a = 9$, $f(9) = 3$. Differentiate $f(x) = x^{1/2}$:

$$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f'(9) = \frac{1}{6}.$$

So $L(x) = 3 + \frac{1}{6}(x - 9)$ and $\sqrt{9.2} \approx 3 + \frac{1}{6}(0.2) = 3 + 0.0333\ldots = 3.0333$. (Choice (D) is the true value, not the tangent-line estimate.) The answer is **(B)**.

Independent 2: Over or under for a logarithm

The linearization of $f(x) = \ln x$ at $a = 1$ is used to approximate $\ln(0.9)$. This approximation is

(A) an overestimate, because f is concave down near $x = 1$

(B) an underestimate, because f is concave up near $x = 1$

(C) exact

(D) an underestimate, because f is decreasing

(E) undefined, because $0.9 < 1$

Solution. The linearization is $L(x) = x - 1$, so $\ln(0.9) \approx -0.1$. Differentiate twice: $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2} < 0$, so f is concave down near $a = 1$ and the graph lies below its tangent line. The tangent-line value is therefore an overestimate. (Indeed $\ln(0.9) = -0.10536\ldots < -0.1$.) The answer is **(A)**.

Independent 3: L'Hospital with a trig numerator

$\lim_{x \to 0} \frac{\sin(5x)}{2x} =$

(A) 0

(B) $\frac{5}{2}$

(C) 5

(D) $\frac{2}{5}$

(E) the limit does not exist

Solution. As $x \to 0$, $\sin(5x) \to 0$ and $2x \to 0$, form $\frac{0}{0}$. By L'Hospital's Rule (chain rule on $\sin(5x)$),

$$\lim_{x\to 0} \frac{\sin(5x)}{2x} = \lim_{x\to 0} \frac{5\cos(5x)}{2} = \frac{5\cos 0}{2} = \frac{5}{2}.$$

The answer is **(B)**.

Independent 4: A ratio of polynomials at infinity

$\lim_{x\to\infty} \dfrac{4x^3+1}{x^3-2x} =$

(A) 0

(B) 1

(C) 2

(D) 4

(E) ∞

Solution. As $x \to \infty$ both top and bottom $\to \infty$, form $\frac{\infty}{\infty}$. Applying L'Hospital's Rule repeatedly until the form resolves,

$$\lim_{x\to\infty} \frac{4x^3+1}{x^3-2x} = \lim_{x\to\infty} \frac{12x^2}{3x^2-2} = \lim_{x\to\infty} \frac{24x}{6x} = \lim_{x\to\infty} \frac{24}{6} = 4.$$

The answer is **(D)**. (Equivalently, equal-degree polynomials give the ratio of leading coefficients $\frac{4}{1} = 4$.)

Independent 5 (FRQ): Mixed forms

Evaluate each limit, naming the indeterminate form at each step.

(a) $\lim_{x\to 0} \dfrac{e^x - 1}{\sin x}$

(b) $\lim_{x\to\infty} \dfrac{x^2}{e^x}$

Solution.

(a) A $\frac{0}{0}$ form.

> As $x \to 0$, $e^x - 1 \to 0$ and $\sin x \to 0$, form $\frac{0}{0}$. By L'Hospital's Rule,
>
> $$\lim_{x\to 0} \frac{e^x - 1}{\sin x} = \lim_{x\to 0} \frac{e^x}{\cos x} = \frac{e^0}{\cos 0} = \frac{1}{1} = 1.$$
>
> The limit is 1.

(b) An $\frac{\infty}{\infty}$ form needing two passes.

> As $x \to \infty$, $x^2 \to \infty$ and $e^x \to \infty$, form $\frac{\infty}{\infty}$. Applying the rule,
>
> $$\lim_{x\to\infty} \frac{x^2}{e^x} = \lim_{x\to\infty} \frac{2x}{e^x},$$
>
> still $\frac{\infty}{\infty}$, so apply it again:
>
> $$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0.$$
>
> The limit is 0; the exponential grows faster than any power of x.

Independent 6 (FRQ): A population estimate in context

A population of bacteria (in millions) is a differentiable function $P(t)$ of time t in hours. At $t = 5$ hours, $P(5) = 300$ and $P'(5) = 24$. For times near $t = 5$, the graph of P is concave up.

(a) Use a tangent-line approximation to estimate $P(5.5)$.

(b) Interpret $P'(5)$ in context, with units.

(c) Is your estimate an overestimate or an underestimate of $P(5.5)$? Justify.

Solution.

(a) Estimating $P(5.5)$.

> With base point $t = 5$,
>
> $$L(t) = P(5) + P'(5)(t - 5) = 300 + 24(t - 5),$$
>
> so
>
> $$P(5.5) \approx L(5.5) = 300 + 24(5.5 - 5) = 300 + 24(0.5) = 312 \text{ million bacteria.}$$

(b) Interpreting $P'(5)$.

> $P'(5) = 24$ is the instantaneous growth rate at $t = 5$ hours: the population is increasing at 24 million bacteria per hour at that instant.

(c) Overestimate or underestimate.

> Near $t = 5$, P is concave up ($P'' > 0$), so its graph lies above the tangent line. The tangent-line estimate sits below the true curve, so 312 million is an **underestimate** of $P(5.5)$. In context: the growth is accelerating, so the population overtakes what the constant-rate tangent line predicts.

AP Calculus BC Module 4C
Linearization and L'Hospital's Rule Overview

Both tools here rest on one fact: near a point, a differentiable curve looks like its tangent line. That gives a quick way to *approximate* values, and it explains why comparing *derivatives* resolves a limit shaped like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Key facts

$$L(x) = f(a) + f'(a)(x - a), \qquad f(x) \approx L(x) \text{ for } x \text{ near } a.$$

$$\text{If } \lim_{x \to a} \frac{f(x)}{g(x)} \text{ is } \tfrac{0}{0} \text{ or } \tfrac{\infty}{\infty}, \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Writing a linearization

Pick a convenient base point a where $f(a)$ and $f'(a)$ are exact; then $f(b) \approx f(a) + f'(a)(b - a)$. Keep the structure: value at the base, plus slope at the base, times the small step.

Overestimate or underestimate (concavity decides)

Concave up near a ($f'' > 0$): graph above the tangent, approximation **underestimates**. Concave down near a ($f'' < 0$): graph below the tangent, approximation **overestimates**.

L'Hospital's Rule

1. Confirm the form is $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by substitution.
2. Differentiate numerator and denominator *separately* (not the quotient rule).
3. Re-check the new ratio; repeat while it is still indeterminate, then substitute.

BC **Looking ahead (BC).**

The linearization is the degree-one Taylor polynomial $T_1(x) = f(a) + f'(a)(x - a)$; Unit 10 adds $\frac{f''(a)}{2}(x - a)^2$ and higher terms, and the sign of that next term is the concavity over/under argument made precise.

Justifying (the language readers reward)

"$f(b) \approx f(a) + f'(a)(b - a) =$ *(value)*, using the tangent slope $f'(a)$."

"The estimate is an *(over/under)*estimate because f is *(concave down/up)* near a, i.e. f'' is *(neg./pos.)* there."

"The form is *($\frac{0}{0}$/$\frac{\infty}{\infty}$)*, so by L'Hospital's Rule the limit equals $\lim \frac{f'}{g'} =$ *(value)*."

Common traps

(1) The base point a is where you *know* f; don't drop the $(b - a)$ step. (2) Over/under comes from *concavity* (f''), never from "increasing/decreasing." (3) L'Hospital differentiates top and bottom *separately*; it is not the quotient rule. (4) Check the form *first*, and again before each repeat; never apply it to a non-indeterminate limit.